In our first attempt at finding an exact solution to Einstein's equations
we will seek spherically symmetric solution(s). This is a useful approach because:
a) It is way easier than "less" symmetric solutions.
b) It is obviously relevant for astrophysical applications, e.g. stars, planets, etc.

Let's warm up with a similar exercise from E&M.

Consider Gauss' law: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

Suppose we want a spherically symmetric solution for $\vec{E}$ in a region where there is
no charge density, i.e. $\rho = 0$, e.g. "outside" of a charged object.

Then we need to solve $\underbrace{\vec{\nabla} \cdot \vec{E} = 0}_{\text{"vacuum" form of Gauss' law}}$ with a spherically symmetric ansatz $\vec{E} = f(r) \hat{r}$

— only depends on $r$

— only points in $\hat{r}$

Then: $\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 f(r) \right) = 0 \Rightarrow f(r) = \frac{k}{r^2}$

To interpret this we use an old trick:
$\int \vec{\nabla} \cdot \vec{E} \, d^3x = \int \frac{\rho}{\varepsilon_0} d^3x$     integrate both sides over the volume of a sphere

$\hookrightarrow \oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$          using the divergence theorem

$4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E = \frac{Q_{enc}}{4\pi \varepsilon_0 r^2} \Rightarrow k = \frac{Q_{enc}}{4\pi \varepsilon_0} \Rightarrow \vec{E} = \frac{Q_{enc}}{4\pi \varepsilon_0 r^2} \hat{r}$

This works when $\rho$ is extended or pointlike. Of course if $\rho$ is extended
with radius $R$, we have to find an "interior" solution for $r < R$.

Notice that solving this so easily was greatly facilitated by choosing the right coordinates.
Of course we could have started w/ any coordinate system we like, but the process would
have been much uglier. Along the way, we might have seen how various coordinate redefinitions
could simplify the process and eventually get ourselves to spherical polar. Of course once
we have a solution in any coordinate system the hard work is done and we can take
the solution (not the differential equation) and transform it to any coordinate system we want.

Turning to Einstein's equations, we would like to find a spherically symmetric solution in a source free region, i.e. $\overline{T}_{\mu\nu} = 0$.

Recall:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \overline{T}_{\mu\nu} \qquad EE$$

$$R_{\mu\nu} = 8\pi G \left( \overline{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \qquad EE \text{ in trace-reverse form}$$

Obviously, the latter form is more useful in this situation since $\overline{T}_{\mu\nu} = 0 \Rightarrow \overline{T} = 0$ and it reduces to:

$$\boxed{R_{\mu\nu} = 0} \qquad (\text{"vacuum" form of } EE)$$

Ultimately we want to solve this for $g_{\mu\nu}(x)$ which contains 10 independent functions:

$$g_{tt}, g_{rr}, g_{\theta\theta}, g_{\phi\phi} \quad g_{rt}, g_{r\theta}, g_{r\phi}, g_{t\theta}, g_{t\phi}, g_{\theta\phi}$$

I have already assumed that we will use something like spherical polar coordinates $\{t, r, \theta, \phi\}$. We will find that this is nonetheless still not the simplest coordinate system in which to find (or even present) a solution. We might expect this since our naive sense of spherical polar is based on flat-space, and we expect our solution to be curved.
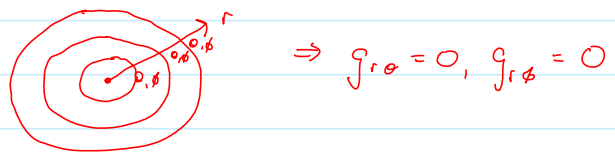
Of course spherical symmetry will have a lot to say about the 10 unknown functions.

One way to impose spherical symmetry is by imagining the spacetime in terms of an $S^2$-foliation. This means we build up the spacetime by stacking concentric 2-spheres (like an onion), except in this case they are stacked along a radial direction r (so moving along r changes the "size" of each $S^2$) and lined up along t.
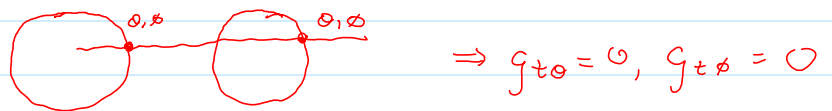
This has some immediate consequences:

a) If we focus on one of the 2-spheres (fix r and t) then the metric should take the form $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \Rightarrow g_{\theta\phi} = 0$ and we know that the rest of the metric components should not depend on $\theta$ or $\phi$.

b) If we align our $S^2$-shells so that radial geodesics go through the same value of $\theta$ and $\phi$ on each slice

$\Rightarrow g_{r\theta} = 0, \; g_{r\phi} = 0$

Additionally, if we stack them along t such that motion purely along t keeps the values of $\theta, \phi$ unchanged

$\Rightarrow g_{t\theta} = 0, \; g_{t\phi} = 0$

So spherical symmetry eliminates 5 of the unknown functions, combine 2 and tells us that nothing else depends on $\theta$ or $\phi$.

So we have: $ds^2 = -A(r,t)dt^2 + 2\underline{B(r,t)dr\,dt} + C(r,t)dr^2 + D(r,t)r^2 d\Omega^2$

Carroll is a bit more careful about this term, but we won't worry about that complication.

So we have 4 unknown functions $A, B, C, D$. But we can say more.

$$\underset{g_{tt}}{\uparrow} \quad \underset{g_{rr}}{\uparrow} \quad \underset{g_{tr}}{\uparrow}$$

Let's redefine $r \to r' = \sqrt{D(r,t)}\, r$   which can be inverted to find $r(r', t)$.

Then: $ds^2 = -A(r',t)\,dt^2 + 2B(r',t)\,dr'\,dt + C(r',t)\,dt^2 + r'^2 d\Omega^2$

This is very similar to choosing a gauge to help simplify solving Maxwell's eqns for $\phi$ and $\vec{A}$.

Assuming this redefinition, we will drop the primes henceforth.

Exploiting this gauge freedom even further consider redefining time w/

$$t \rightarrow t' = t - f(r, t') \qquad \Rightarrow \qquad t = t' + f(r, t')$$

Then:
$$dt = dt' + \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial t'} dt'$$
$$= \frac{\partial f}{\partial r} dr + \left(1 + \frac{\partial f}{\partial t'}\right) dt'$$
$$dt^2 = \left(\frac{\partial f}{\partial r}\right)^2 dr^2 + 2 \frac{\partial f}{\partial r}\left(1 + \frac{\partial f}{\partial t'}\right) dr dt' + \left(1 + \frac{\partial f}{\partial t'}\right)^2 dt'^2$$

And putting this back into the metric we find:
$$ds^2 = -A(r, t')\left[\left(\frac{\partial f}{\partial r}\right)^2 dr^2 + \left(1 + \frac{\partial f}{\partial t'}\right)^2 dt'^2 + 2\frac{\partial f}{\partial r}\left(1 + \frac{\partial f}{\partial t'}\right) dr dt'\right]$$
$$+ 2B(r, t')\left[\frac{\partial f}{\partial r} dr^2 + \left(1 + \frac{\partial f}{\partial t'}\right) dr dt'\right]$$
$$+ C(r, t') dr^2 + r^2 d\Omega^2$$

Collecting the $dr dt'$ cross-terms:
$$2\left[-A(r, t')\frac{\partial f}{\partial r} + B(r, t')\right]\left(1 + \frac{\partial f}{\partial t'}\right) dr dt'$$
$$= 0 \quad \text{for } f(r, t') \text{ such that } \frac{\partial f}{\partial r} = \frac{B(r, t')}{A(r, t')} \quad \Rightarrow \quad f(r, t') = \int \frac{B(r, t')}{A(r, t')} + g(t')$$

Assuming that we redefine $t \rightarrow t'$ using just such a function $f(r, t')$ then the metric becomes:
$$ds^2 = -\left[A(r, t')\left(1 + \frac{\partial f}{\partial t'}\right)^2\right] dt'^2$$
$$+ \left[-A(r, t')\left(\frac{B(r, t')}{A(r, t')}\right)^2 + 2B(r, t')\frac{B(r, t')}{A(r, t')} + C(r, t')\right] dr^2$$
$$+ r^2 d\Omega^2$$

Dropping the primes this is essentially:
$$ds^2 = \underbrace{g(r, t)}_{g_{tt}} dt^2 + \underbrace{g(r, t)}_{g_{rr}} dr^2 + r^2 d\Omega^2$$

2 unknown functions!

But we can say even more about these unknown functions $g_{tt}, g_{rr}$.

If we assume $A(r,t), C(r,t) > 0$  [that the original metric was Lorentzian] then we can conclude:

$g_{tt} < 0$

$g_{rr} = -\dfrac{B^2}{A} + 2\dfrac{B^2}{A} + C = \dfrac{B^2}{A} + C > 0$     w/out any assumption for $B(r,t)$

To encode what we know about the signs into the unknown functions we can write:

$g_{tt} = -e^{2\alpha(r,t)}$

$g_{rr} = e^{2\beta(r,t)}$

Everything so far has only used spherical symmetry, and gauge (coordinate) choices. We haven't even used the Einstein equation $R_{\mu\nu} = 0$!

Taking our metric so far and cranking out $R_{\mu\nu}$, we find that most terms are already $0$, and the only nontrivial ones are $R_{tt}, R_{rr}, R_{tr}, R_{\theta\theta}, R_{\phi\phi}$. To satisfy EE, we must make sure that these nontrivial terms vanish as well and this will further shape our solution.

Consider first:

$R_{tr} = \frac{2}{r} \frac{\partial \beta}{\partial t} = 0 \qquad \Rightarrow \quad \beta(r,t) = \beta(r)$

then

$R_{\theta\theta} = e^{-2\beta} \left[ r \left( \frac{\partial \beta}{\partial r} - \frac{\partial \alpha}{\partial r} \right) - 1 \right] + 1 = 0$

$\Downarrow$

$\partial_t R_{\theta\theta} = -2 \cancelto{0}{\frac{\partial \beta}{\partial t}} e^{-2\beta} \left[ \quad \right] + e^{-2\beta} \left[ r \cancelto{0}{\frac{\partial^2 \beta}{\partial t \partial r}} - r \frac{\partial^2 \alpha}{\partial t \partial r} \right] = 0$

hence

$-r e^{-2\beta} \partial_t \partial_r \alpha(r,t) = 0 \qquad \Rightarrow \quad \alpha(r,t) = f(r) + g(t)$

Then:

$ds^2 = -e^{2f(r)} e^{2g(t)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$

But now we redefine $t \to t' = \int e^{g(t)} dt \quad \Rightarrow dt' = e^{g(t)} dt$ to get:

$ds^2 = -e^{2f(r)} dt'^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$       Hey guys! I'm diagonal :)

Notice anything unusual? None of the metric components depend on $t'$!
That is, without assuming time-independence, we were able to show that the metric (in an appropriate set of coordinates) does not depend on time, i.e. $g_{\mu\nu}(r,t) = g_{\mu\nu}(r)$.

Such a space is called stationary (not changing). It also implies the existence of a timelike Killing vector $\partial_t$.

But this space is even gooder, because it is also invariant under $t \to -t$ due to the absence of any $dx^i dt$ cross-terms. This extra condition makes the space static (not doing anything).

To understand the difference, consider a planet which is sitting still vs. one that is spinning. The former would create a static geometry while the latter would only create a stationary geometry, since $t \to -t$ reverses the spin.

Thus our assumption of a spherically symmetric source free solution implies a static geometry.

To finish up we need the actual form of $f(r)$ and $\beta(r)$.
Returning to $R_{\mu\nu} = 0$ w/ everything we know so far:

$$R_{tt} = e^{2(f-\beta)} \left[ \frac{\partial^2 f}{\partial r^2} + \left(\frac{\partial f}{\partial r}\right)^2 - \frac{\partial f}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial f}{\partial r} \right] = 0 \quad \Rightarrow \quad [\ ] = 0$$

$$R_{rr} = -\frac{\partial^2 f}{\partial r^2} - \left(\frac{\partial f}{\partial r}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial \beta}{\partial r} = 0$$

But: $\underset{0}{[\ ]} + \underset{0}{R_{rr}} = \frac{2}{r}\left(\frac{\partial f}{\partial r} + \frac{\partial \beta}{\partial r}\right) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial r} = -\frac{\partial \beta}{\partial r}$

$$\Downarrow$$

$$f(r) = -\beta(r) + c$$

Then:

$$ds^2 = -e^{-2\beta(r)} \overbrace{e^{2c} dt^2}^{= dt^2 \text{ after redefining } t} + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

Finally:

$$R_{\theta\theta} = e^{2f}\left(-2r\frac{\partial f}{\partial r} - 1\right) + 1 = 0$$

or $\frac{\partial}{\partial r}\left(r e^{2f}\right) = 1 \quad \Rightarrow \quad r e^{2f} = r + c$

$$e^{2f} = 1 + \frac{c}{r} \quad \Rightarrow \quad e^{2\beta} = \left(1 + \frac{c}{r}\right)^{-1}$$

So:

$$ds^2 = -\left(1 + \frac{c}{r}\right)dt^2 + \left(1 + \frac{c}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

But we know from the Newtonian limit that $g_{00} = -(1 + 2\phi) = -(1 - \frac{2GM}{r})$

so finally:

$$\phi = -\frac{GM}{r}$$

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

This is the Schwarzschild geometry, expressed in Schwarzschild coordinates. This is the geometry outside of any spherically symmetric mass, e.g. planet, star, black hole.

We call $R = 2GM$ the Schwarzschild radius.

For $r \to \infty$ or $M \to 0$   $ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2 \simeq \mathbb{M}^4$ as expected!